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ON THE STABILITY OF AUTONOMOUS SYSTEMS IN THE PRESENCE OF SEVERAL RESONANCES

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The equilibrium position of an autonomous system of ordinary differential equations is investigated in the critical case of n pairs of pure imaginary roots in the simultaneous presence of several resonance relationships. Sufficient conditions of stability and instability of the equilibrium position in the first nonlinear order are derived for a special class of systems. Results are extended to Hamiltonian systems. Oscillations of a satellite about the position of relative equilibrium on a circular orbit is considered as an example.

1. Let us consider the autonomous system of ordinary differential equations

$$x_*^{\cdot} = Ax_* + X_*(x_*), \quad x_*^{\cdot} = dx_* / dt \tag{1.1}$$

$$x_* = (x_1^*, \ldots, x_{2n}^*), \quad X_* = (X_1^*, \ldots, X_{2n}^*), \quad X_*(0) = 0$$

where x_* and X_* are $2n$-dimensional vectors of the Euclidean space E_{2n}, A is a constant square matrix with only pure imaginary eigenvalues $\pm i\omega_s$ ($\omega_s > 0$, $s = 1, \ldots, n$) among which there are no multiple eigenvalues, and $X_s^*(x_*)$ are holomorphic functions whose expansions in powers of x_* begin with m-th order forms.

Let system (1.1) have $\mu > 1$ resonance relationships of the form

$$\langle \Omega, P_\nu \rangle = 0, \quad \nu = 1, \ldots, \mu \tag{1.2}$$

$$\Omega = (\omega_1, \ldots, \omega_q), \quad P_\nu = (p_{\nu 1}, \ldots, p_{\nu q})$$

$$|P_\nu| = \sum_{j=1}^{q} |p_{\nu j}| = k, \quad \sum_{j=1}^{q} \Big| \sum_{\nu=1}^{\mu} \varkappa_\nu p_{\nu j} \Big| > k, \quad k = m + 1 \geqslant 3$$

where P_ν is a vector of dimension q ($q \leqslant n$) with integral relatively prime components, $\varkappa_\nu$ ($\nu = 1, \ldots, \mu$) are arbitrary integral constants at least two of which are nonzero, and k is an odd number.

The case of simultaneous presence of several resonance relationships was previously considered in [1 —5].

It was shown in [6] that using the nondegenerate complex linear transformation it is possible to reduce system (1.1) to the form

$$x^{\cdot} = i\omega x + \sum_{l=m\geqslant 2}^{\infty} X^{(l)}(x, y), \quad y^{\cdot} = -i\omega y + \sum_{l=m\geqslant 2}^{\infty} Y^{(l)}(x, y) \tag{1.3}$$

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n), \quad \omega = \{\omega_1, \ldots, \omega_n\}$$

where x and y are complex conjugate vectors, ω is a diagonal matrix, and $X^{(l)}$ and $Y^{(l)}$ are complex conjugate vector functions whose components $X_s^{(l)}$ and $Y_s^{(l)}$ ($s = 1, \ldots, n$) are l-th order forms of x and y.

Subjecting system (1.3) to a number of successive transformations described in [7] and taking into account (1.2), we obtain [5] in polar coordinates r_s, φ_s ($s = 1, \ldots, n$) the following normal form of system (1.3) accurate to the first nonlinear terms:

$$r_j^{\cdot} = 2\sum_{\nu=1}^{\mu} R_\nu Q_{\nu j}(\theta_\nu) + O(\|r\|^{(k+1)/2}) \tag{1.4}$$

$$\theta_\nu^{\cdot} = \sum_{i=1}^{\mu}\sum_{j=1}^{q} \frac{|p_{\nu j}|}{r_j} R_i Q_{ij}'(\theta_i) + O(\|r\|^{(k-1)/2})$$

$$r_\alpha^{\cdot} = O(\|r\|^{(k+1)/2}), \quad \varphi_\alpha^{\cdot} = O(\|r\|^{(k-1)/2})$$

$$j = 1, \ldots, q; \quad \nu = 1, \ldots, \mu; \quad \alpha = q+1, \ldots, n$$

$$R_\nu^2 = \prod_{l=1}^{q} r_l^{|p_{\nu l}|}, \quad \theta_\nu = \sum_{j=1}^{q} |p_{\nu j}| \varphi_j, \quad r = (r_1, \ldots, r_n)$$

$$Q_{\nu j}(\theta_\nu) = a_{\nu j}\cos\theta_\nu + b_{\nu j}\sin\theta_\nu, \quad Q_{\nu j}' = dQ_{\nu j} \mid d\theta_\nu$$

$$(Q_{\nu j}(\theta_\nu) \equiv 0, \quad \text{if} \quad p_{\nu j} = 0)$$

Let us consider the class of model systems (obtainable from (1.4) by rejecting the omitted terms) for which

$$a_{\nu j} = 0, \; b_{\nu j} \neq 0, \; \text{if} \quad p_{\nu j} \neq 0, \; \nu = 1, \ldots, \mu; \; j = 1, \ldots, q$$

so that

$$r_j^{\cdot} = 2\sum_{\nu=1}^{\mu} b_{\nu j} R_\nu \sin\theta_\nu, \quad \theta_\nu^{\cdot} = \sum_{i=1}^{\mu}\sum_{j=1}^{q} \frac{b_{ij}|p_{\nu j}|}{r_j} R_i \cos\theta_i \tag{1.5}$$

$$j = 1, \ldots, q; \quad \nu = 1, \ldots, \mu$$

(only the resonance subsystem appears above).

We shall investigate the equilibrium position of the model system (1.5) with the resonance relationships (1.2) satisfied.

Theorem 1.1. For the model system (1.5) to have an increasing solution of the type of the invariant beam

$$r_j = k_j b(t), \quad k_j > 0, \quad j = 1, \ldots, q. \tag{1.6}$$

$$\theta_\xi = \pi/2, \; \xi = 1, \ldots, \mu_0; \; \theta_\eta = -\pi/2, \; \eta = \mu_0 + 1, \ldots, \mu$$

$$(0 \leqslant \mu_0 \leqslant \mu)$$

it is necessary and sufficient that

$$k_j = \frac{H_{1j} - H_{2j}}{H_{11} - H_{21}} k_1, \quad H_{1j} > H_{2j}, \quad H_{1j} = \sum_{\xi=1}^{\mu_0} h_{\xi j}, \quad H_{2j} = \sum_{\eta=\mu_0+1}^{\mu} h_{\eta j} \tag{1.7}$$

$$b_{\nu j} = h_{\nu j} / R_\nu^\circ, \quad R_\nu^\circ = R_\nu(k_1, \ldots, k_q) \qquad (1.8)$$
$$j = 1, \ldots, q; \quad \nu = 1, \ldots, \mu$$

where $h_{\nu j}$ are some constants.

P r o o f. Substituting a solution of the form (1.6) into system (1.5) we obtain

$$k_j b^\cdot = 2\Sigma_j b^{k/2}, \quad \Sigma_j = \sum_{\xi=1}^{\mu_0} b_{\xi j} R_\xi^\circ - \sum_{\eta=\mu_0+1}^{\mu} b_{\eta j} R_\eta^\circ, \quad j = 1, \ldots, q$$

Evidently solution (1.6) of system (1.5) exists, if such $k_j > 0$ can be found that

$$k_j = (\Sigma_j / \Sigma_1) k_1, \quad \Sigma_j / 0, \quad j = 1, \ldots, q \qquad (1.9)$$

Using (1.8) we express relationships (1.9) in the form (1.7).

The converse statement is proved similarly.

T h e o r e m 1 . 2. For the model system (1.5) to have an unstable partiticular solution of the type of increasing beam

$$r_u = k_u b(t), \quad k_u > 0, \quad u = 1, \ldots, \bar{q}$$
$$r_v = 0, \quad v = \bar{q} + 1, \ldots, q \quad (0 < \bar{q} < q)$$
$$\theta_\xi = \pi / 2, \ \xi = 1, \ldots, \mu_0; \quad \theta_\eta = -\pi / 2, \quad \eta = \mu_0 + 1, \ldots, \bar{\mu}$$
$$\theta_\zeta = \pm \pi/2, \ \zeta = \bar{\mu} + 1, \quad \ldots, \quad \mu \ (0 \leqslant \mu_0 \leqslant \bar{\mu} < \mu; \ \bar{\mu} > 0)$$

it is necessary and sufficient that $b_{\nu u}$ ($\nu = 1, \ldots, \bar{\mu}$; $u = 1, \ldots, \bar{q}$) satisfy conditions (1.7) and (1.8), and that in addition

$$p_{\nu v} = 0, \quad \nu = 1, \ldots, \bar{\mu}; \quad v = \bar{q} + 1, \ldots, q$$

$$\sum_{v=\bar{q}+1}^{q} |p_{\zeta v}| > 1, \quad \zeta = \bar{\mu} + 1, \ldots, \mu$$

Proof of this is omitted owing to its simplicity.

These results may be transferred, with some obvious alterations, to the class of model systems (1.4) for which

$$a_{\nu j} \neq 0, \quad b_{\nu j} = 0, \quad \text{if} \quad p_{\nu j} \neq 0, \ \nu = 1, \ldots, \mu; \ j = 1, \ldots, q$$

2. Let us consider in more detail the problem of stability of the equilibrium position of the Hamiltonian system

$$x_s^\cdot = \frac{\partial H(x, y)}{\partial y_s}, \quad y_s^\cdot = -\frac{\partial H(x, y)}{\partial x_s}, \quad s = 1, \ldots, n \qquad (2.1)$$

when it cannot be solved in linear approximation and the resonance relationships (1.2) are valid.

Let us assume that the Hamiltonian of system (2.1) is of the form

$$H(x, y) = H_2(x, y) + H_k(x, y) + H_{k+1}(x, y) + \ldots \qquad (2.2)$$

$$H_2(x, y) = \frac{1}{2} \sum_{s=1}^{n} (-1)^{\delta_s} (x_s^2 + \omega_s^2 y_s^2)$$

where δ_s is either unity or two (see, e.g., [8]) and $H_l(x, y)$ is a homogeneous polynomial of power l.

We introduce the notation

$$\lambda_s = (-1)^{\delta_s}\omega_s, \quad s = 1, \ldots, n$$

$$p_{\nu j}^* = (-1)^{\delta_s} p_{\nu j}, \quad \nu = 1, \ldots, \mu; \ j = 1, \ldots, q$$

Using the canonical polynomial transformation with allowance for (1.2) we can reduce the Hamiltonian (2.2) to the normal form up to the k-th order. Rejecting terms of order higher than the k-th, we represent the Hamiltonian of the model system in canonical polar variables as follows:

$$\Gamma = \sum_{s=1}^{n} \lambda_s r_s + 2 \sum_{\nu=1}^{\mu} A_\nu \Big(\prod_{l=1}^{q} r_l^{|p_{\nu l}^*|}\Big)^{1/2} \cos\Big(\sum_{j=1}^{q} p_{\nu j}^* \varphi_j\Big) \tag{2.3}$$

The system of equations that corresponds to (2.3) is of the form

$$r_j^{\cdot} = -2\sum_{\nu=1}^{\mu} A_\nu p_{\nu j}^* R_\nu \sin\theta_\nu^*, \quad j = 1, \ldots, q \tag{2.4}$$

$$\theta_\nu^{*\cdot} = -\sum_{i=}^{\mu}\sum_{j=1}^{q} A_i \frac{p_{\nu j}^* | p_{ij}^* |}{r_j} R_i \cos\theta_i^*, \quad \nu = 1, \ldots, \mu$$

$$\theta_\nu^* = \sum_{j=1}^{q} p_{\nu j}^* \varphi_j$$

Below we assume that $A_\nu \neq 0$ $(\nu = 1, \ldots, \mu)$.

T h e o r e m 2 . 1. For the Hamiltonian system (2.4) to have an increasing solution of the type of invariant beam

$$r_j = k_j b(t), \quad k_j > 0, \quad j = 1, \ldots, q$$

$$\theta_\xi^* = (\pi / 2)\,\mathrm{sign}\, A_\xi, \quad \xi = 1, \ldots, \mu_0$$

$$\theta_\eta^* = -(\pi / 2)\,\mathrm{sign}\, A_\eta, \quad \eta = \mu_0 + 1, \ldots, \mu \quad (0 \leqslant \mu_0 \leqslant \mu)$$

it is necessary and sufficient that

$$k_j = \frac{\langle P_{1j}^*, G_1\rangle - \langle P_{2j}^*, G_2\rangle}{\langle P_{11}^*, G_1\rangle - \langle P_{21}^*, G_2\rangle} k_1 \tag{2.5}$$

$$\langle P_{2j}^*, G_2\rangle - \langle P_{1j}^*, G_1\rangle > 0, \quad j = 1, \ldots, q$$

$$| A_\nu | = g_\nu / R_\nu^\circ, \quad \nu = 1, \ldots, \mu \tag{2.6}$$

$$P_{1j}^* = (p_{1j}^*, \ldots, p_{\mu_0 j}^*), \quad P_{2j}^* = (p_{\mu_0+1, j}^*, \ldots, p_{\mu j}^*)$$

$$G_1 = (g_1, \ldots, g_{\mu_0}), \quad G_2 = (g_{\mu_0+1}, \ldots, g_\mu), \quad g_\nu > 0$$

where $P_{\varepsilon j}^*$, G_ε $(\varepsilon = 1,2)$ are vectors of dimension μ_0 when $\varepsilon = 1$ and $\mu - \mu_0$ when $\varepsilon = 2$, and g_ν are some constants.

The proof is similar to that of Theorem 1.1.

T h e o r e m 2 . 2. For the Hamiltonian system (2.4) to have an unstable particular solution of the type of increasing beam

$$
\begin{aligned}
& r_u = k_u b(t), \quad k_u > 0, \quad u = 1, \ldots, \bar{q} \\
& r_v = 0, \quad v = \bar{q} + 1, \ldots, q \ (0 < \bar{q} < q) \\
& \theta_\xi^* = (\pi / 2) \operatorname{sign} A_\xi, \quad \xi = 1, \ldots, \mu_0 \\
& \theta_\eta^* = -(\pi / 2) \operatorname{sign} A_\eta, \quad \eta = \mu_0 + 1, \ldots, \bar{\mu} \\
& \theta_\zeta^* = \pm \pi / 2, \quad \zeta = \bar{\mu} + 1, \ldots, \mu \ (0 \leqslant \mu_0 \leqslant \bar{\mu} < \mu; \ \bar{\mu} > 0)
\end{aligned}
$$

it is necessary and sufficient that $p_{\nu u}^*$ and A_ν ($\nu = 1, \ldots, \bar{\mu}$; $u = 1, \ldots, \bar{q}$) satisfy conditions (2.5) and (2.6), respectively, and that in addition

$$p_{\nu v}^* = 0, \quad \nu = 1, \ldots, \bar{\mu}; \ v = \bar{q} + 1, \ldots, q$$

$$\sum_{v=\bar{q}+1}^{q} |p_{\zeta v}^*| > 1, \quad \zeta = \bar{\mu} + 1, \ldots, \mu$$

We use the notation

$$P^* = \| p_{\nu j}^* \|, \quad \nu = 1, \ldots, \mu; \quad j = 1, \ldots, q$$

$$C_{q_0} = \operatorname{col} \{c_1, \ldots, c_{q_0}, 0, \ldots, 0\}$$

where C_{q_0} is an arbitrary column vector of dimension q, $q_0 \leqslant q$.

Theorem 2.3. For the Hamiltonian system (2.4) to have an integral of the form

$$\sum_{\gamma=1}^{q_0} c_\gamma r_\gamma + \sum_{\alpha=q+1}^{n} r_\alpha = \text{const} \tag{2.7}$$

it is necessary and sufficient that the equality

$$P^* C_{q_0} = 0 \tag{2.8}$$

is satisfied.

Proof. Equating to zero the derivative of (2.7) we obtain on the strength of the model system (2.4) a system of equations which must be satisfied by the constants c_γ

$$\sum_{\gamma=1}^{q_0} p_{\nu\gamma}^* c_\gamma = 0, \quad \nu = 1, \ldots, \mu$$

The completion of proof is evident.

A similar theorem is valid also for the model system (1.5) with the substitution of the equality [9]

$$B C_{q_0} = 0, \quad B = \| b_{\nu j} \|$$

Corollaries. 1°. If vector C_{q_0} with positive components $c_j > 0$ which satisfies equality (2.8) exists, the equilibrium position of the model Hamiltonian system (2.4) is stable with respect to variables $r_1, \ldots, r_{q_0}, r_{q+1}, \ldots, r_n$ [10].

2°. If the model Hamiltonian system (2.4) has two weak resonances (see [5]) of the third order, the equilibrium position of that system is stable.

3°. If the model Hamiltonian system (2.4) has three weak resonances of the third order, the equilibrium position of that system is stable.

Example. Let us consider the problem of oscillations of a satellite about the relative equilibrium position on a circular orbit. It was shown in [11] that in suitable canonical variables the Hamiltonian of the linearized system of the considered problem reduces to the form

$$H_2 = -\tfrac{1}{2}(\xi_1^2 + \omega_1^2\eta_1^2) + \tfrac{1}{2}(\xi_2^2 + \omega_2^2\eta_2^2) + \tfrac{1}{2}(\xi_3^2 + \omega_3^2\eta_3^2)$$

and that the double resonance

$$\omega_3 - 2\omega_1 = 0, \quad \omega_3 - \omega_2 + \omega_1 = 0$$

may be realized here.

The corresponding model system expressed in canonical polar coordinates is of the form

$$r_j^{\cdot} = \alpha_j A_1 \sqrt{r_1^2 r_3} \sin\theta_1^* + \beta_j A_2 \sqrt{r_1 r_2 r_3} \sin\theta_2^*, \quad j = 1,2,3 \tag{2.9}$$
$$\alpha_1 = -4,\ \alpha_2 = 0,\ \alpha_3 = -2,\ \beta_1 = \beta_2 = 2,\ \beta_3 = -2$$
$$\theta_1^* = 2\varphi_1 + \varphi_3, \quad \theta_2^* = \varphi_3 - \varphi_2 - \varphi_1$$

(the equations for θ_ν^* are omitted here).

By Theorem 2.1 system (2.9) has an increasing solution of the form

$$r_j = k_j b(t), \quad k_j > 0, \quad j = 1,2,3$$
$$\theta_\nu^* = (-1)^\nu (\pi/2)\ \mathrm{sign}\ A_\nu, \quad \nu = 1,2$$

if

$$k_2 = \frac{g_2}{2g_1 + g_2} k_1, \quad k_3 = \frac{g_1 - g_2}{2g_1 + g_2} k_1, \quad g_1 > g_2$$

$$|A_1| = g_1 / \sqrt{k_1^2 k_3}, \quad |A_2| = g_2 / \sqrt{k_1 k_2 k_3}$$

Setting $|A_2 / A_1| = A$, after some transformations we obtain the condition $0 < A < \sqrt{3}$. Hence, when this inequality is satisfied, the equilibrium position of the model system (2.9) is unstable.

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